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A linear theory of rotating, thermally stratified, hydromagnetic flow

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The hydromagnetic flow of a thermally stratified fluid confined between two rotating parallel plates is studied. The flow is assumed to be linear, steady and axially symmetric. The flow is driven both mechanically and thermally and general thermal boundary conditions are applied. Attention is focused upon the mechanism controlling the interior fluid (diffusion, Ekman pumping or hydromagnetic forces) and upon the conditions under which laminated flow ($\partial v/\partial z \neq 0$) may occur. It is found that the occurrence of laminated flow is very sensitive to the thermal boundary conditions and is suppressed by hydromagnetic effects. For mixed boundary conditions, hydromagnetic forces control the interior and laminated flow is suppressed if $\alpha \geqslant O(1)$, where α^2 represents the ratio of hydromagnetic to Coriolis forces. For a constant heat flux, this occurs for a much weaker magnetic field: if $\alpha \geqslant O(E^{\frac{1}{4}})$. For a restricted range of the parameters, a new boundary layer, called the thermomagnetic layer, in which Coriolis, thermal and hydromagnetic forces balance may occur.

1. Introduction

In the past few years there has been a great deal of interest in rotating, thermally stratified flows and in rotating hydromagnetic flows. This interest has been spurred in part by possible applications to geophysical and astrophysical problems and in part by the desire to increase our basic understanding of such flows. The primary purpose of the present paper is to combine these effects and to perform a study of rotating, thermally stratified, hydromagnetic flows. In particular, the ability of the magnetic forces to alter the flow which occurs in the non-magnetic case will be studied.

Chandrasekhar (1961) has studied the thermal stability of confined rotating hydromagnetic fluids and found that hydromagnetic effects have the ability to decrease significantly the stability of such fluids. The present study of stably stratified fluids (heated from above) is, in a sense, the counterpart of Chandrasekhar's study of unstably stratified fluids (heated from below). The studies of wave motions which may occur in unbounded, rotating, stably stratified, hydromagnetic flow (e.g. see Roberts & Soward 1972; Acheson & Hide 1973) do not bear directly upon the present work.

In the absence of stratification and magnetic effects, it is well known that steady flow of a rotating fluid obeys the Taylor–Proudman theorem outside viscous regions and exhibits columnar behaviour (i.e. $\partial/\partial z = 0$, where $\mathbf{\Omega} = \Omega\hat{\mathbf{z}}$).

This constraint is a direct consequence of the vigorous meridional circulation which occurs between regions of differing vorticity in a homogeneous fluid. When this circulation occurs between an interior region and an Ekman layer, it is referred to as Ekman pumping.

One of the most interesting features of thermal stratification is that the Taylor–Proudman theorem gives way to the thermal-wind equation and *laminated* flow may occur. Laminated flow means that $\partial v/\partial z \neq 0$, where v is the azimuthal velocity component and z is the axial co-ordinate (see example 1 of Barcilon & Pedlosky 1967*a*). This may occur when the stratification is sufficiently strong to keep the meridional circulation induced within the Ekman layers from penetrating the stratified interior fluid. In this case, the interior is controlled by diffusion of heat and/or momentum or by Eddington–Sweet currents.

It is well known from Gilman & Benton (1968) and from Loper & Benton (1970) that hydromagnetic forces act in concert with Ekman pumping to control the interior of a homogeneous rotating fluid. These hydromagnetic forces are generated by a meridional circulation of electric current between the interior and the Ekman–Hartmann layer. This circulation of current, called a Hartmann current, is strikingly similar in form and effect to Ekman pumping. In particular, hydromagnetic forces act to maintain columnar behaviour in the fluid just as the rotational constraint does [see equation (3.11) below].

In the combined problem, there are three effects vying for control of the interior fluid: diffusion, Ekman pumping and Hartmann currents. In the following analysis, we shall investigate the conditions under which magnetic effects control the interior.

In order to simplify this presentation as much as possible, only the simplest geometry and boundary conditions will be used and the problem will be linearized. Steady flow in a radially unbounded cylindrical geometry will be studied, allowing a von Kármán type similarity to be employed. However, the flow will be driven both mechanically, by differential rotation of the boundaries, and thermally, by a centrifugal buoyancy term, and general thermal boundary conditions will be applied.

The problem is formulated, non-dimensionalized and linearized in § 2. A similarity transformation is introduced in § 3. It is noted in this section that some assumption is crucial for the similarity to be successful and that the nature of this assumption governs, in part, whether laminated flow may occur or not. Also in § 3, a hydromagnetic version of the thermal-wind equation is derived and discussed in detail. Solutions of the problem are presented and discussed in §§ 4 and 5. Several limiting cases are presented in § 4 to allow comparison with known results. Finally, a summary and conclusions are presented in § 6.

2. Formulation of the problem

Let us consider a thermally stratified, quasi-incompressible fluid with constant properties confined between two flat rotating boundaries. The fluid is electrically conducting, the boundaries are insulators and a uniform axial magnetic field is applied. We wish to study steady, axially symmetric, linearized flow in which heating by viscous dissipation is negligible. We begin with the following set of governing equations, written in a rotating co-ordinate system:

$$\rho(\mathbf{q}\,.\nabla)\mathbf{q}+2\rho\Omega\hat{\mathbf{k}}\times\mathbf{q} = -\nabla p+\rho\nabla(\tfrac{1}{2}\Omega^2r^2) -\rho g\hat{\mathbf{k}}+\mu_m^{-1}(\nabla\times\mathbf{B})\times\mathbf{B}+\nu\rho\nabla^2\mathbf{q}, \tag{2.1}$$

$$\nabla\,.\mathbf{q} = 0, \quad (\mathbf{q}\,.\nabla)T = \kappa\nabla^2T, \qquad (2.2),\ (2.3)$$

$$\rho = \rho_0[1-\bar{\alpha}(T-T_0)], \tag{2.4}$$

$$\nabla\times(\mathbf{B}\times\mathbf{q}) = (\mu_m\bar{\sigma})^{-1}\nabla^2\mathbf{B}, \quad \nabla\,.\mathbf{B} = 0, \qquad (2.5),\ (2.6)$$

where $\mathbf{q}, \mathbf{B}, p, T, \rho, g, \nu, \mu_m, \bar{\sigma}, \kappa, \bar{\alpha}, r, \Omega$ and $\hat{\mathbf{k}}$ are, respectively, the fluid velocity vector, magnetic-field vector, pressure, temperature, density, acceleration due to gravity, kinematic viscosity, magnetic permeability, electrical conductivity, thermal diffusivity, thermal expansion, cylindrical radius, rotation rate and an axial unit vector.

The governing equations of motion may be cast in dimensionless form by assuming

$$\left.\begin{aligned} \mathbf{q} &= \epsilon\Omega L\mathbf{q}^*, \quad T = T_0+(\Delta T/L)z+(\epsilon\Omega^2L/\bar{\alpha}g)T^*, \\ \rho &= \rho_0[1-(\bar{\alpha}\Delta T/L)z-(\epsilon\Omega^2L/g)T^*], \\ p &= p_0+\tfrac{1}{2}\rho_0\Omega^2r^2-\rho_0gz+\tfrac{1}{2}(\rho_0g\bar{\alpha}\Delta T/L)z^2+\epsilon\rho_0\Omega^2L^2p^*, \\ \mathbf{B} &= B_0(\hat{\mathbf{k}}+\epsilon\bar{\sigma}\mu_m\Omega L^2\mathbf{B}^*), \\ \nabla &= \nabla^*/L, \quad z = Lz^*, \quad r = Lr^*, \end{aligned}\right\} \tag{2.7}$$

where L is the distance between the two bounding plates, T_0 is the temperature of the bottom plate in the absence of motion and $T_0+\Delta T$ is the temperature of the top plate in the absence of motion. The parameter ϵ measures the strength of the fluid flow generated by motion of the boundary and an asterisk denotes a dimensionless variable. This scaling assumes that Eddington–Sweet currents are not larger than motion induced by the boundaries. [See comments following equation (4.6).]

With axisymmetric flow, the solenoidal equations (2.2) and (2.6) are identically satisfied by the introduction of a stream function and a current function:

$$\left.\begin{aligned} \mathbf{q}^* &= \psi^*_{z^*}\hat{\imath}+v^*\hat{\jmath}-r^{*-1}(r^*\psi^*)_{r^*}\hat{\mathbf{k}}, \\ \mathbf{B}^* &= \phi^*_{z^*}\hat{\imath}+b^*\hat{\jmath}-r^{*-1}(r^*\phi^*)_{r^*}\hat{\mathbf{k}}, \end{aligned}\right\} \tag{2.8}$$

where $\hat{\imath}$ and $\hat{\jmath}$ are radial and azimuthal unit vectors and a subscript denotes differentiation.

To linearize the governing equations, we must assume that

$$\bar{\alpha}\Delta T \ll 1, \quad \epsilon \ll 1, \quad \epsilon\bar{\sigma}\mu_m\Omega L^2 \ll 1, \quad \epsilon\sigma\psi\delta \ll E, \quad \epsilon\Omega^2L/g \ll 1, \tag{2.9}$$

where δ represents the scale of variation of the stream function ψ. (In the interior $\delta = 1$; in the Ekman layer $\delta = E^{\frac{1}{2}}$.) The dimensionless equations of motion may now be linearized and written in component form (dropping the asterisks) as

$$-2v = -p_r-SFrz/\epsilon-FrT+2\alpha^2(\nabla^2-r^{-2})\phi+E(\nabla^2-r^{-2})\psi_z, \tag{2.10}$$

$$2\psi_z = 2\alpha^2 b_z + E(\nabla^2 - r^{-2})v, \tag{2.11}$$

$$0 = -p_z + T - E\nabla^2[(r\psi)_r/r], \tag{2.12}$$

$$-(\sigma S/r)(r\psi)_r = E\nabla^2 T, \tag{2.13}$$

$$-\psi_{zz} = (\nabla^2 - r^{-2})\phi_z, \quad -v_z = (\nabla^2 - r^{-2})b, \qquad (2.14), (2.15)$$

$$-(r\psi_z)_r/r = \nabla^2[(r\phi)_r/r], \tag{2.16}$$

where
$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{\partial^2}{\partial z^2},$$

$$E = \nu/\Omega L^2, \quad \sigma = \nu/\kappa, \quad F = \Omega^2 L/g, \quad S = \bar{\alpha}\Delta T g/\Omega^2 L, \quad \alpha^2 = \bar{\sigma}B_0^2/2\rho_0\Omega. \tag{2.17}$$

In this study, we shall ignore the vertical side walls and assume that the fluid is unbounded radially. One reason for this simplification is the fact that the side-wall boundary layers are certain to present a formidable problem. In fact, a correct analysis of the side-wall boundary layers in a homogeneous rotating electrically conducting fluid has not yet been published. [Ingham's (1969) analysis is incorrect; his side-wall layers cannot carry the required vertical flux of electric current. A correct analysis has recently been completed by Vempaty (1974).] A second reason for ignoring the side walls is the fact that, for a large range of the parameters, they play only a passive role. Also, for the parameter range in which the side walls actively control the interior, the present analysis surprisingly yields the correct result, as will be shown by comparison with Barcilon & Pedlosky (1967*b*). However, it should be noted that it is possible to obtain similarity solutions which cannot be matched to any side-wall boundary layers (e.g. see Barcilon & Pedlosky 1967*c*), so the relevance of the present results to a confined geometry is not assured.

With the neglect of the side walls, it is sufficient to specify boundary conditions at $z = 0$ and $z = 1$. We shall assume that the bounding plates are rigid rotators, the bottom one being at rest in the rotating system and the top one rotating differentially with angular velocity $\epsilon\Omega$. The boundaries are assumed to be electrical insulators. The analysis will be performed simultaneously for two types of thermal boundary condition which yield differing solutions: case *A* satisfies constant-heat-flux conditions (Neumann conditions) while case *B* satisfies mixed thermal boundary conditions. Altogether we specify

with
$$\left.\begin{array}{l} \psi = \psi_z = b = \phi_z = 0, \quad v = (\tfrac{1}{2} \pm \tfrac{1}{2})r \\ T_z = 0 \quad (\text{case } A) \quad \text{or} \quad T + CT_z = 0 \quad (\text{case } B) \end{array}\right\} \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2},$$

where the constant C is known.

The problem may be considerably simplified by ignoring the viscous terms in (2.10)–(2.12) and replacing the above boundary conditions by the steady Ekman–Hartmann compatibility conditions (see the appendix):

with
$$\left.\begin{array}{l} \psi = mE^{\frac{1}{2}}(-\tfrac{1}{2}r \mp \tfrac{1}{2}r \pm v) \\ b = nE^{\frac{1}{2}}(\tfrac{1}{2}r \pm \tfrac{1}{2}r \mp v), \quad \phi_z = \dfrac{\alpha^2 n - m}{(\alpha^4+1)^{\frac{1}{2}}}E^{\frac{1}{2}}(\tfrac{1}{2}r \pm \tfrac{1}{2}r - v) \\ T_z = 0 \quad (\text{case } A) \quad \text{or} \quad T + CT_z = 0 \quad (\text{case } B) \end{array}\right\} \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}, \tag{2.18}$$

where
$$n, m = \left[\frac{(\alpha^4+1)^{\frac{1}{2}} \pm \alpha^2}{4(\alpha^4+1)}\right]^{\frac{1}{2}}. \tag{2.19}$$

3. Development of a similarity solution

Since the ordinary hydrodynamic solution (Greenspan & Howard 1963), the hydromagnetic solution (Loper & Benton 1970) and the stratified solution (Barcilon & Pedlosky 1967*b*) all display the von Kármán type of similarity outside the side-wall boundary layers, it is reasonable to assume that the variables in the present combined analysis exhibit the same similarity:

$$\left.\begin{aligned} v(r,z) = r\omega(z), \quad \psi(r,z) = r\chi(z),\\ p(r,z) = \rho(z)+\tfrac{1}{2}r^2\zeta(z), \quad b(r,z) = r\delta(z),\\ T(r,z) = \eta(z)+\tfrac{1}{2}r^2\tau(z). \end{aligned}\right\} \tag{3.1}$$

Note that the temperature field has a term proportional to r^2. Its absence precludes consideration of laminated flow as can be seen from the thermal-wind equation $\partial v/\partial z = \partial T/\partial r$.

Equations (2.10)–(2.13) and (2.15) now become

$$2\alpha^2\chi' - 2\omega + \zeta + F\eta + SFz/\epsilon = -\tfrac{1}{2}r^2F\tau, \tag{3.2}$$

$$\chi' - \alpha^2\delta' = 0, \tag{3.3}$$

$$\rho' - \eta = \tfrac{1}{2}r^2(-\zeta' + \tau), \tag{3.4}$$

$$-2\sigma S\chi - E\eta'' - 2E\tau = \tfrac{1}{2}r^2E\tau'', \tag{3.5}$$

$$\omega' + \delta'' = 0, \tag{3.6}$$

where the variable ϕ has been eliminated by use of (2.16) and a prime denotes differentiation with respect to z.

Boundary conditions (2.18) now are

$$\left.\begin{aligned} &\chi = mE^{\frac{1}{2}}(-\tfrac{1}{2} \mp \tfrac{1}{2} \pm \omega), \quad \delta = nE^{\frac{1}{2}}(\tfrac{1}{2} \pm \tfrac{1}{2} \mp \omega)\\ \text{with} \quad &\tau_z = \eta_z = 0 \quad (\text{case } A)\\ \text{or} \quad &\tau + C\tau_z = \eta + C\eta_z = 0 \quad (\text{case } B) \end{aligned}\right\} \text{ at } z = \tfrac{1}{2} \pm \tfrac{1}{2}. \tag{3.7}$$

In order that the similarity transformation be successful, all radial dependence must be eliminated from the problem. This means that (3.4) and (3.5) must each be divided into two equations by separately equating terms involving like powers of the radius to zero. Also, the term on the right-hand side of (3.2) must either be set equal to zero or assumed to be negligibly small. This must be done with care since the manner in which this troublesome term is neglected will affect the form of the solution. It may be neglected by assuming F to be sufficiently small (see Duncan 1966; Barcilon & Pedlosky 1967*a*; Niimi 1971) or by assuming $\tau = 0$ (see Barcilon & Pedlosky 1967*c*; Homsy & Hudson 1969). Either of these assumptions allows mechanically driven laminated flow to occur. However, if it is assumed that both F is small and $\tau = 0$ (see Sakurai 1969*b*) it is impossible to obtain mechanically driven laminated flow without side-wall effects. In what follows, we shall assume that F is sufficiently small in case A to allow neglect of the right-hand side of (3.2) compared with the dominant terms on the left-hand side:

$$r^2F\tau \ll (\omega, \zeta, \alpha^2\chi', SF/\epsilon, F\eta) \quad (\text{case } A). \tag{3.8}$$

This, in effect, places a restriction upon the radial range of validity of the solution. For case B, no restriction upon F is necessary since $\tau = 0$, as will be seen below.

A number of papers in the literature (Duncan 1966; Barcilon & Pedlosky 1967a, b; Sakurai 1969a, b; Niimi 1971) have assumed F to be sufficiently small to allow neglect of all terms in the radial momentum equation involving F. This precludes the study of thermally driven fluid motions (unless explicitly forced by the thermal boundary conditions as in Duncan 1966). Also, neglect of these terms at this point makes it difficult to determine the range of F for which the resulting solution is valid. To help clarify these matters the terms $F\eta$ and SFz/ϵ will be retained in (3.2).

The right-hand side of (3.5) reveals that τ is, at most, a linear function of z. Application of the boundary conditions on τ leads to τ = constant for case A and $\tau = 0$ for case B. This is in agreement with Niimi (1971), who found that it is necessary to impose a constant-heat-flux boundary condition to permit a non-trivial value of τ to exist. Note that the term $E\tau$ in (3.5) represents radial conduction of heat. It appears that this component of the thermal field would, in a more complete problem, be forced by the side-wall boundary conditions. In the present problem, the value of τ is determined by a compatibility condition on the similarity equations. That is, from the left-hand side of (3.5), we find

$$\tau = \begin{cases} -\dfrac{\sigma S}{E}\displaystyle\int_0^1 \chi(z)\, dz & (\text{case } A), \\ 0 & (\text{case } B), \end{cases} \tag{3.9}$$

where the boundary conditions on η have been used.

The variable ρ appears only on the left-hand side of (3.4) and will not be considered further. The variable ζ may be eliminated between the left-hand side of (3.2) and the right-hand side of (3.4) to yield

$$2\alpha^2\chi'' - 2\omega' + \tau + F\eta' + SF/\epsilon = 0. \tag{3.10}$$

The problem now consists of (3.3), the left-hand side of (3.5), (3.6) and (3.10) subject to conditions (3.7), where τ is given by (3.9).

A hydromagnetic version of the thermal-wind equation may be obtained by differentiation of (3.3) and use of (3.6) and (3.10):

$$2(1+\alpha^4)\omega' = SF/\epsilon + \tau + F\eta'. \tag{3.11}$$

ω' is a measure of the strength of the lamination of the flow. In the absence of stratification ($S = 0$), it is readily found from (3.9) and (3.5) that $\tau = \eta' = 0$ and (3.11) reduces to $\omega' = 0$ whether magnetic forces are present or not. That is, in the absence of stratification, magnetic forces are incapable of producing laminated flow.

With stratification, the three terms on the right-hand side of (3.11) are capable of driving laminated flow. The first term arises from a centrifugal buoyancy force and is capable of driving an Eddington–Sweet circulation. These thermally driven flows have been investigated by Barcilon & Pedlosky (1967*c*) and Homsy & Hudson (1969). The second term arises from a vertical buoyancy force and is generated either by explicit forcing in the thermal boundary conditions (Duncan 1966) or indirectly via the mechanical boundary conditions (Barcilon & Pedlosky 1967*a*, *b*; Niimi 1971). In all previous analyses, the third term has been absent or has been assumed small. It will be seen in § 5 that this term may become important in a new horizontal boundary layer.

The ability of hydromagnetic effects to suppress laminated flow may be clearly seen in (3.11). As the magnetic interaction parameter α increases above unity, the magnitude of the laminated flow decreases rapidly, as the fourth power of α. In addition to this obvious transition at $\alpha = O(1)$, another occurs at a much weaker magnetic field: $\alpha = O(E^{\frac{1}{4}})$. This may be seen as follows. Assuming $\omega' = O(1)$, (3.6) yields $\delta'' = O(1)$ and (3.3) yields $\chi' = O(\alpha^2)$. Thus if $\alpha^2 \geqslant O(E^{\frac{1}{2}})$ the meridional circulation is no longer suppressed by stratification but is at least as strong as in the unstratified case.† It will be seen below in § 4 that laminated flow is suppressed whenever $\alpha \gg E^{\frac{1}{4}}$ for the case of constant heat flux (case A) and whenever $\alpha \gg 1$ for the case of mixed thermal boundary conditions (case B).

Differentiation of (3.10) and (3.11) and combination with the left-hand side of (3.5) yield a single equation for χ:

$$\chi'''' - \lambda^3\chi = E\tau\lambda^3/\sigma S, \tag{3.12}$$

where

$$\lambda^3 \equiv \sigma S F\alpha^2/E(1+\alpha^4). \tag{3.13}$$

The form of the solution depends upon the magnitude of this new parameter λ. If $\lambda \ll 1$, the problem has simple polynomial solutions, which are presented and discussed in § 4. All known results (i.e. for either $S = 0$ or $\alpha = 0$) have $\lambda = 0$. If $\lambda \gg 1$, entirely new boundary layers occur which involve both thermal and hydromagnetic effects. These solutions are presented in § 5.

4. Solutions valid for $\lambda \ll 1$

If $\lambda \ll 1$, the solutions of (3.3), the left-hand side of (3.5), (3.6) and (3.10) subject to conditions (3.7) for cases A and B (corresponding to $i = A$ and B respectively) are

$$\omega_i = \tfrac{1}{2} + K_i(z-\tfrac{1}{2}), \tag{4.1}$$

$$\chi_i = -\tfrac{1}{2}mE^{\frac{1}{2}} + \tfrac{1}{2}K_i[mE^{\frac{1}{2}} + \alpha^2(z-z^2)], \tag{4.2}$$

$$\delta_i = \tfrac{1}{2}nE^{\frac{1}{2}} + \tfrac{1}{2}K_i[-nE^{\frac{1}{2}} + z - z^2], \tag{4.3}$$

$$\tau_A = \tfrac{1}{2}mE^{-\frac{1}{2}}\sigma S - \tfrac{1}{2}\sigma S E^{-1}K_A[mE^{\frac{1}{2}} + \tfrac{1}{6}\alpha^2], \tag{4.4a}$$

$$\tau_B = 0, \tag{4.4b}$$

$$\eta_A = \eta_0 + \tfrac{1}{12}\sigma S\alpha^2 E^{-1}K_A z^2(1-z)^2, \tag{4.5a}$$

† If we assume $\sigma S = O(1)$, as in Barcilon & Pedlosky (1967*a*), this argument shows that their analysis must include hydromagnetic effects if $\alpha^2 \geqslant O(E)$.

$$\eta_B = \tfrac{1}{2}\sigma SmE^{-\frac{1}{2}}(1-K_B)(2C^2+C-2Cz-z+z^2)$$
$$+\lambda^3(S/24\epsilon)(-2C^2-C+2Cz+z-2z^3+z^4), \qquad (4.5b)$$

$$K_A = \frac{\sigma SE^{-\frac{1}{2}}[m+2FE^{\frac{1}{2}}/\sigma\epsilon]}{4(1+\alpha^4)+\sigma SE^{-\frac{1}{2}}[m+\alpha^2/6E^{\frac{1}{2}}]}, \qquad (4.6a)$$

$$K_B = SF/2\epsilon(1+\alpha^4). \qquad (4.6b)$$

The parameters K_A and K_B measure the strength of the lamination and are of particular interest in the subsequent analysis. In (4.6a), the first term in the square bracket in the numerator represents mechanically driven laminated flow and the second thermally driven laminated flow. It may be seen from (4.6b) that there is no mechanically driven laminated flow for case B. If we wish to study thermally driven flow without differential rotation of the boundaries, all terms in (4.1)–(4.6) not divided by ϵ should be omitted. Then we may set $\epsilon = SF = \bar{\alpha}\Delta T$ without loss of generality.

The above solutions contain many effects: rotation, stratification, hydromagnetic forces and thermal and mechanical driving. In order to increase our confidence in and understanding of these solutions, we shall investigate several limiting cases.

4.1. *Unstratified flow* ($S = 0$)

In this limit, the lamination constants K_A and K_B are zero and the solutions reduce to

$$\omega = \tfrac{1}{2}, \quad \chi = -\tfrac{1}{2}mE^{\frac{1}{2}}, \quad \delta = \tfrac{1}{2}nE^{\frac{1}{2}}, \quad \tau = 0. \qquad (4.7\,a\text{–}d)$$

Equation (4.7) yields the well-known result that the angular speed of the interior fluid is the average of the speeds of the bounding plates. This holds true for both ordinary hydrodynamic flow and hydromagnetic flow. The functions χ and δ are proportional to the axial velocity and axial electric current, respectively. If $\alpha = 0$, $\chi = -\frac{1}{4}E^{\frac{1}{2}}$ ($w = \frac{1}{2}E^{\frac{1}{2}}$) in agreement with equation (2.17.10) of Greenspan (1968). Also, (4.7b) and (4.7c) agree with equations (31) and (37) of Gilman & Benton (1968).

4.2. *Non-magnetic flow* ($\alpha = 0$)

In this limit, the solutions reduce to

$$\omega_i = \tfrac{1}{2}+K_i(z-\tfrac{1}{2}), \quad \chi_i = \tfrac{1}{4}(K_i-1)E^{\frac{1}{2}}, \qquad (4.8), (4.9)$$

$$\tau_A = (1-K_A)\sigma S/4E^{\frac{1}{2}}, \qquad (4.10)$$

where $$K_A = \frac{\sigma S/E^{\frac{1}{2}}+4FS/\epsilon}{8+\sigma S/E^{\frac{1}{2}}}, \quad K_B = SF/2\epsilon. \qquad (4.11\,a,b)$$

It is apparent from (4.11 a) that the thermally driven motions are small compared with those driven mechanically provided that

$$F \ll \epsilon\sigma/4E^{\frac{1}{2}}.$$

If $F \ll \epsilon\sigma/4E^{\frac{1}{2}}$, the solution (4.11 a) agrees with equation (21) of Niimi (1971), which is not surprising since he also adopted a similarity approach. However, in this case, (4.11 a) also agrees with Barcilon & Pedlosky (1967b; see footnote on

p. 619) and this is surprising for the following reason. They obtained their solution by solving an interior flow problem controlled by viscosity and thermal diffusion and did not assume a similarity like (3.1) *a priori*. However, their solutions in the footnote on p. 619 satisfy $\nabla^2 v = \nabla^2 w = 0$ if we assume V_B and V_T to be linear in r. Thus the viscous forces are solenoidal and the interior must be controlled by radial diffusion of heat which generates a thermal field proportional to r^2 identical to that given by (4.10a).

The laminated flow driven by the centrifugal buoyancy term in (4.10a) is indistinguishable in form from that driven by the mechanical forcing. The two effects produce laminated flows of equal magnitude whenever $F = O(\epsilon\sigma E^{-\frac{1}{2}})$. In fact, if $\epsilon = -4FE^{\frac{1}{2}}/\sigma$, the two effects cancel and no lamination occurs.

For case B, the lamination predicted by (4.8) and (4.11b) is in agreement with equation (3.7a) of Barcilon & Pedlosky (1967c) and with equation (3.3) of Homsy & Hudson (1969). In this case, the lamination is produced entirely by the centrifugal buoyancy term; there is no mechanically driven lamination.

4.3. *Stratified, weakly magnetic flow*

For case A, it is sufficient to investigate the effect of the magnetic forces upon the stratification in the limit of a weak magnetic field ($\alpha \ll 1$) since the non-magnetic balance is easily upset for this case. The lamination constant K_A, given by (4.6a), becomes

$$K_A = \frac{\sigma S E^{-\frac{1}{2}}[1+4FE^{\frac{1}{2}}/\sigma\epsilon]}{8+\sigma S E^{-\frac{1}{2}}[1+\alpha^2/3E^{\frac{1}{2}}]}. \tag{4.12}$$

It is easy to see from (4.12) that K_A is strongly diminished whenever $\alpha \gg E^{\frac{1}{4}}$. That is, *for the constant-heat-flux boundary condition* (case A), *laminated flow is suppressed whenever* $\alpha \gg E^{\frac{1}{4}}$, *regardless of the strength of the stratification*. Assuming $1 \gg \alpha \gg E^{\frac{1}{4}}$ and strong stratification ($\sigma S \gg E^{\frac{1}{2}}$), the solutions are, to leading order,

$$\omega_A = \tfrac{1}{2} + (z-\tfrac{1}{2})\,12FE/\sigma\epsilon\alpha^2, \tag{4.13}$$

$$\chi_A = \tfrac{1}{4}E^{\frac{1}{2}}(-1+6z-6z^2) + (6z-6z^2)\,FE/\sigma\epsilon, \tag{4.14}$$

$$\delta_A = \tfrac{1}{4}E^{\frac{1}{2}} + (z-z^2)\,(3E^{\frac{1}{2}}/2\alpha^2 + 6FE/\sigma\epsilon\alpha^2), \tag{4.15}$$

$$\tau_A = -3\sigma S/4\alpha^2 - FS/\epsilon, \tag{4.16}$$

$$\eta_A = \eta_0 + K_A z^2(1-z^2)\,\sigma S\alpha^2/12E. \tag{4.17}$$

In this limit, mechanically forced laminated flow is suppressed while the thermal forcing SF/ϵ drives a much smaller laminated flow, of order

$$\left(\frac{SF}{\epsilon}\frac{E^{\frac{1}{2}}}{\sigma S}\frac{E^{\frac{1}{2}}}{\alpha^2}\right).$$

This lack of lamination occurs because, to leading order, $\tau + SF/\epsilon = 0$ and, with the assumption $\lambda \ll 1$, the term $F\eta'$ is small [see (3.11)]. It is interesting that, in the strongly stratified limit, meridional circulation is no longer suppressed but is driven by the hydromagnetic forces in (3.3). The advection of heat by χ in (3.5) is balanced partly by radial conduction τ and partly by vertical conduction η''.

Now condition (3.8) becomes $r^2F \ll 1$, which requires $F \ll 1$ for the radial range of validity to be large. $F \ll 1$ is consistent with the assumption that λ is small.

It is apparent from (4.5*b*) that laminated flow for case B is suppressed by hydromagnetic effects only for $\alpha \gg 1$. In this case, $\tau = 0$ and it is impossible for the terms τ and SF/ϵ to cancel in (3.11) as in case A.

4.4. *Strong magnetic field* ($\alpha \gg 1$)

In this limit, the solutions simplify to

$$\omega_i = \tfrac{1}{2} + K_i(z-\tfrac{1}{2}), \tag{4.18}$$

$$\chi_i = -E^{\frac{1}{2}}/2^{\frac{3}{2}}\alpha^3 + \tfrac{1}{2}\alpha^2 K_i(z-z^2), \tag{4.19}$$

$$\delta_i = E^{\frac{1}{2}}/2^{\frac{3}{2}}\alpha + \tfrac{1}{2}K_i(z-z^2), \tag{4.20}$$

$$\tau_A = \sigma S/2^{\frac{3}{2}}\alpha^3 E^{\frac{1}{2}} - K_A\alpha^2\sigma S/12E, \tag{4.21}$$

where

$$K_A = \frac{3\sigma S E^{-\frac{1}{2}}[2^{-\frac{1}{2}} + 4FE^{\frac{1}{2}}\alpha^3/\sigma\epsilon]}{\alpha^5(24\alpha^2 E + \sigma S)}, \tag{4.22a}$$

$$K_B = SF/2\epsilon\alpha^4. \tag{4.22b}$$

From (4.22*a*) it may be seen that, for case A, the lamination decreases at least as fast as α^{-2} for $\alpha \gg 1$ and, for certain parameter ranges, as fast as α^{-7}. Also the lamination driven by the thermal term SF/ϵ rapidly becomes more important than that driven mechanically as α increases beyond unity.

Case B is much less complicated: as α increases beyond unity, laminated flow decreases as α^{-4} as predicted by (3.11).

5. Solutions valid for $\lambda \gg 1$

In this limit, the region outside the Ekman–Hartmann layers splits into three: an interior, a top boundary layer and a bottom boundary layer. Since (3.12) is a third-order equation, the upper and lower layers do not have identical structure: the upper layer behaves like $\exp[-\lambda(1-z)]$ while the lower layer behaves like $\exp(-\tfrac{1}{2}\lambda z)\sin(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z + k)$. The upper layer is thin and monotonic while the lower layer is thicker and oscillatory. Since these layers rely on both thermal and magnetic forces for their existence, they will be called *thermomagnetic* layers, or TM layers for brevity.

In this section we shall assume that σ, S, α and F are all of unit order when ordered with respect to E while ϵ is small. This makes λ large, of order $E^{-\frac{1}{4}}$. In what follows, only the leading-order solutions will be given.

The solutions to the problem for case A are

$$\omega = \tfrac{1}{2} + R_A\lambda\alpha^{-2}\{\exp[-\lambda(1-z)] - 2\exp(-\tfrac{1}{2}\lambda z)\cos(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z)\}, \tag{5.1}$$

$$\chi = EF/\sigma\epsilon + R_A\{-\exp[-\lambda(1-z)] + \exp(-\tfrac{1}{2}\lambda z)[-\cos(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z) + 3^{\frac{1}{2}}\sin(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z)]\}, \tag{5.2}$$

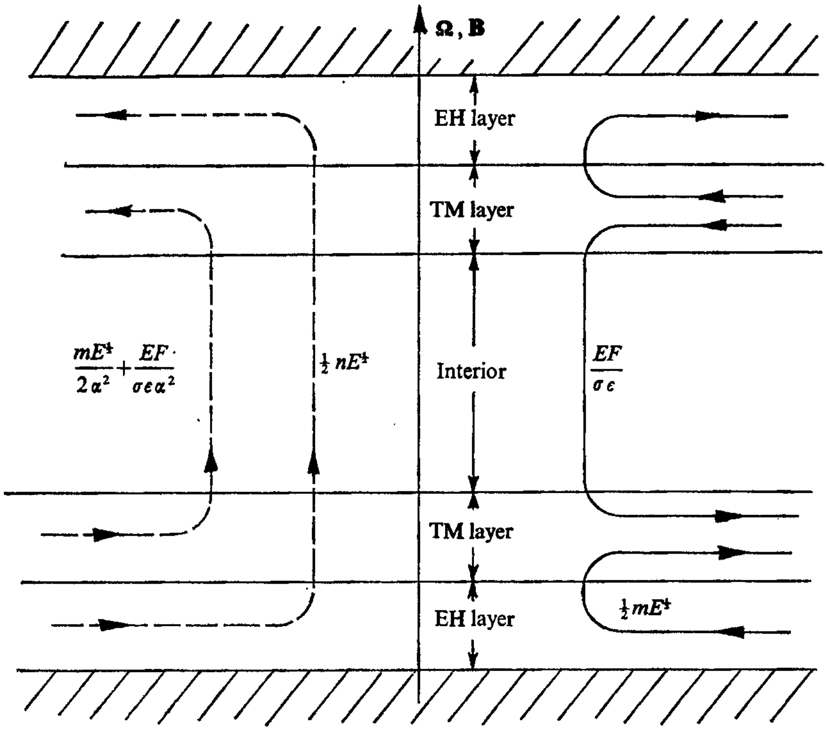

FIGURE 1. ——, schematic meridional topology of lines tangential to velocity field; – – –, electric current.

$$\delta = \tfrac{1}{2}nE^{\frac{1}{2}} + R_A\alpha^{-2}\{1-\exp[-\lambda(1-z)] + \exp(-\tfrac{1}{2}\lambda z)[-\cos(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z) + 3^{\frac{1}{2}}\sin(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z)]\}, \tag{5.3}$$

$$\eta = \eta_0 + 2\sigma S R_A E^{-1}\lambda^{-2}\{-\lambda z + \exp[-\lambda(1-z)] - 2\exp(-\tfrac{1}{2}\lambda z)\cos(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z)\}, \tag{5.4}$$

$$\tau = -SF/\epsilon + 2\sigma SFR_A/\epsilon\lambda, \tag{5.5}$$

where
$$R_A = \tfrac{1}{2}mE^{\frac{1}{2}} + EF/\sigma\epsilon. \tag{5.6}$$

For case B, to leading order,

$$\omega = R_B\{-1 + mE^{\frac{1}{2}}\lambda\alpha^{-2}\exp[-\lambda(1-z)] + \exp(-\tfrac{1}{2}\lambda z)[-2\cos(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z) + (2\times 3^{-\frac{1}{2}})\sin[\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z)]\}, \tag{5.7}$$

$$\chi = R_B\{-mE^{\frac{1}{2}}\exp[-\lambda(1-z)] + (4\alpha^2/3^{\frac{1}{2}}\lambda)\exp(-\tfrac{1}{2}\lambda z)\sin(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z)\}, \tag{5.8}$$

$$\delta = R_B\alpha^{-2}\{(m-n\alpha^2)E^{\frac{1}{2}} - mE^{\frac{1}{2}}\exp[-\lambda(1-z)] + (4\alpha^2/3^{\frac{1}{2}}\lambda)\exp(-\tfrac{1}{2}\lambda z)\sin(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z)\}, \tag{5.9}$$

$$\eta = (1+C)S/\epsilon - Sz/\epsilon + 2\sigma Sm R_B E^{-\frac{1}{2}}\lambda^{-2}\exp[-\lambda(1-z)] + S\exp(-\tfrac{1}{2}\lambda z)[-\cos(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z) + 3^{-\frac{1}{2}}\sin(\tfrac{1}{2}\times 3^{\frac{1}{2}}\lambda z)]/\epsilon(1-\lambda C), \tag{5.10}$$

where
$$R_B = SF/4\epsilon(1+\alpha^4)(1-\lambda C). \tag{5.11}$$

These solutions are entirely new; there are no limit checks which can be made to compare with previous work. For case A, the solutions appear to be well behaved for the parameter range chosen in (5.1). In the interior, laminated flow is absent to exponential order and the fluid's angular velocity is the average of those of the top and bottom plates, which is reassuring. The angular velocity generated in the TM layers is small. However, since these layers are much thicker than the EH (Ekman–Hartmann) layers, this slight angular-velocity defect is able to drive a meridional circulation of the same order as that due to the EH layers. (This is quite similar to the situation in the magnetic diffusion region for a homogeneous fluid; see Benton & Loper 1969.)

The mass flows $\frac{1}{2}mE^{\frac{1}{2}}$ pumped by the top and bottom EH layers close within the respective TM layers and do not penetrate the interior. However, this does not mean that vertical motions are absent from the interior; a circulation of strength $EF/\sigma\epsilon$ is drawn radially inwards within the top TM layer, then moves downwards across the interior and radially outwards within the bottom TM layer (see figure 1). The heat advected downwards by this vertical motion is balanced by radial conduction of heat (that is, the terms $2\sigma S\chi$ and $E\tau$ in (3.5) balance within the interior). This same value of τ generates a vertical pressure gradient ζ', via the right-hand side of (3.4), which is just that value required to balance the centrifugal buoyancy term SFz/ϵ in (3.2).

The electric current $\frac{1}{2}nE^{\frac{1}{2}}$ pumped by the EH layers completes a circuit as it would in the unstratified case. In addition, there is a thermally induced circulation of magnitude $mE^{\frac{1}{2}}/2\alpha^2 + EF/\sigma\epsilon\alpha^2$ which is drawn radially inwards within the lower TM layer, then moves upwards across the interior and radially outwards within the top TM layer (see figure 1). These vertical currents which pass through the interior do not exert any forces on the fluid if $\omega' = 0$. If lamination is present ($\omega' \neq 0$) these currents act to suppress it [see (3.6)].

The thermal perturbations τ and η are larger than unit order for the parameter range chosen but they are still small compared with the imposed thermal field, so that the linearization is still valid.

In contrast to case A, the solutions for $\lambda \gg 1$ with mixed thermal boundary conditions (case B) are pathological with large perturbations and unexpected features such as the bottom TM layer being much stronger than the top. It may be seen from (5.10) that the order of the induced thermal field η is the same as that applied and the linearization is invalid. In fact, the total temperature field is isothermal in the interior; all heat transferred must flow through the bottom TM layer. This situation is very similar to that discussed by Barcilon & Pedlosky (1967c) in their § 5. We must conclude, as they did, that it is extremely unlikely that solutions (5.7)–(5.11) can represent any flow within a finite cylinder.

6. Summary and conclusions

This has been a study of steady, linear, axisymmetric, rotating, thermally stratified, hydromagnetic flow in a radially unbounded cylinder. The flow was driven both mechanically (by differential rotation of the boundaries) and thermally (by a centrifugal buoyancy force). The analysis has been performed for both constant-heat-flux boundary conditions (case A) and mixed thermal boundary conditions (case B).

The primary goal of this study was to determine the factors which cause or prevent laminated flow (i.e. $\partial v/\partial z \neq 0$). The factors found to be most important are the thermal boundary conditions and the magnitude of the magnetic interaction parameter α. The results, summarized in dimensional terms, are as follows. For constant-heat-flux boundary conditions (case A) with $\sigma SF\alpha^2/E(1+\alpha^4) \ll 1$,

$$\frac{\partial v}{\partial z} = \Omega \frac{m\sigma S\epsilon E^{-\frac{1}{2}} + 2SF}{\sigma SE^{-\frac{1}{2}}[m+\alpha^2/6E^{\frac{1}{2}}]+4(1+\alpha^4)} \frac{r}{L}.$$

For mixed thermal boundary conditions (case B) with $\sigma SF\alpha^2/E(1+\alpha^4) \ll 1$,

$$\partial v/\partial z = r\Omega SF/2L(1+\alpha^4).$$

For constant-heat-flux boundary conditions (case A) with $\sigma SF\alpha^2/E(1+\alpha^4) \gg 1$,

$$\partial v/\partial z = 0.$$

The nature of the solutions and the influence of the governing parameters are quite different depending upon the thermal boundary conditions chosen. For Dirichlet or mixed boundary conditions (case B), the temperature is completely independent of the radius ($\tau = 0$), while for Neumann conditions (case A), a portion τ of the thermal field may vary as r^2 and radial conduction of heat is possible. Consequently, laminated flow may be driven both thermally and mechanically for case A but only thermally for case B. Thermally driven motions are much smaller than mechanically driven ones if $F \ll m\epsilon\sigma/4E^{\frac{1}{2}}$.

It was found that magnetic effects control the interior and suppress laminated flow for $\alpha \gg E^{\frac{1}{4}}$ for case A and for $\alpha \gg 1$ for case B. For case A, the magnetic influence upon the magnitude of the radial conduction τ causes the right-hand side of (3.11) to become very small whenever $\alpha \gg E^{\frac{1}{4}}$, thus suppressing laminated flow. On the other hand, for case B, $\tau = 0$ and cannot balance SF/ϵ in (3.11); in this case the factor $1+\alpha^4$ on the left-hand side acts to suppress laminated flow.

One unexpected development in the analysis was the appearance of a new type of boundary layer, dubbed a thermomagnetic (TM) layer, which occurs for

$$\sigma SF\alpha^2/E(1+\alpha^4) \gg 1.$$

Within this layer a combination of thermal, magnetic and Coriolis forces balances, although the last are not essential for the existence of the layer. The solutions for this layer for case A appear to be quite reasonable, showing a complete suppression of laminated flow in the interior. The solutions for case B violate the assumptions made to linearize the problem and therefore cannot be trusted.

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Appendix

The purpose of this appendix is to derive the hydromagnetic analogue of the Ekman compatibility conditions, referred to as the Ekman–Hartmann compatibility conditions in a thermally stratified fluid. This analysis will be more general than is necessary for the main text in order to provide proper perspective. We begin by assuming that each dependent variable consists of interior and boundary-layer parts, e.g.

$$v = v^i + v^{\pm},$$

where the superscripts i, $+$ and $-$ denote components in the interior and top and bottom Ekman–Hartmann layers respectively. To analyse the top and bottom layers simultaneously, we may introduce the same stretched co-ordinate for both: let

$$z = \tfrac{1}{2} \pm \tfrac{1}{2} \mp E^{\frac{1}{2}}\zeta, \tag{A 1}$$

where the upper and lower signs refer to the top and bottom layers, respectively. Now

$$\partial(\ )^{\pm}/\partial z = \mp E^{-\frac{1}{2}}\,\partial(\ )^{\pm}/\partial\zeta.$$

In writing down the equations for the Ekman–Hartmann layers, care must be exercised in deriving the proper thermal equation. The linearized form (2.13) is misleading because the neglected terms contain axial derivatives which are large in the boundary layers; we need to start with the nonlinear form

$$-\sigma S(r\psi)_r + \sigma\epsilon(r\psi_z T_r - (r\psi)_r T_z) = Er\nabla^2 T. \tag{A 2}$$

In the main text, the nonlinear terms have been neglected by assuming $\epsilon\sigma\psi\delta \ll E$, where δ is the boundary-layer thickness. It should be noted that the present analysis with an imposed thermal field independent of the radius differs from that of Carrier (1965) with an imposed field constant on paraboloidal surfaces $z - \frac{1}{2}Fr^2$.

The boundary-layer equations derived from (2.10)–(2.12), (A 2) and (2.14)–(2.16) are

$$-2v^{\pm} = -p^{\pm}_r - F_r T^{\pm} \pm 2\alpha^2 E^{-\frac{1}{2}}\psi^{\pm}_{\zeta} \mp E^{-\frac{1}{2}}\psi^{\pm}_{\zeta\zeta\zeta}, \tag{A 3}$$

$$2\psi^{\pm} = 2\alpha^2 b^{\pm} \mp E^{\frac{1}{2}} v^{\pm}_{\zeta}, \tag{A 4}$$

$$0 = \pm E^{-\frac{1}{2}} p^{\pm}_{\zeta} + T^{\pm} - (r\psi^{\pm}_{\zeta\zeta})_r/r, \tag{A 5}$$

$$-\sigma S(r\psi^{\pm})_r + \sigma\epsilon E^{-\frac{1}{2}}[\mp r\psi^{\pm}_{\zeta}(T^i_r + T^{\pm}_r) \pm (r\psi^i + r\psi^{\pm})_r T^{\pm}_{\zeta}] = T^{\pm}_{\zeta\zeta}, \tag{A 6}$$

$$b^{\pm}_{\zeta} = \pm E^{\frac{1}{2}} v^{\pm}, \quad \phi^{\pm}_{\zeta} = \pm E^{\frac{1}{2}}\psi^{\pm}. \qquad \text{(A 7), (A 8)}$$

Equation (A 6) is nonlinear. Assuming that

$$\epsilon\sigma\psi^i \ll E^{\frac{1}{2}}, \quad \epsilon\sigma\psi^{\pm} \ll E^{\frac{1}{2}}, \tag{A 9}$$

it reduces to a linear form:

$$-\sigma S(r\psi^{\pm})_r \mp \sigma\epsilon E^{-\frac{1}{2}} r T^i_r \psi^{\pm}_{\zeta} = T^{\pm}_{\zeta\zeta}. \tag{A 6a}$$

The centrifugal buoyancy term in (A 3) is negligibly small provided that

$$FrT^{\pm} \ll E^{-\frac{1}{2}}\psi^{\pm}_{\zeta\zeta\zeta}, \tag{A 10}$$

and we obtain the dynamics for linear Ekman–Hartmann layers. If constraint (A 10) were not satisfied, we should have a buoyant boundary layer; non-magnetic forms of such a layer have been studied by Carrier (1965) and Hsueh (1969).

We now turn to the boundary conditions. Assuming the boundaries to be impermeable and electrically insulating but allowing general azimuthal velocity and thermal boundary conditions yields

$$\left.\begin{array}{r} \psi = \psi_z = b = \phi_z = 0 \\ v = v_\pm(r), \quad AT + CT_z = 0 \end{array}\right\} \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}.$$

In boundary-layer notation these conditions are

$$\left.\begin{array}{c} \psi^i + \psi^\pm = 0, \quad \psi^i_z \mp E^{-\frac{1}{2}} \psi^\pm_\zeta = 0 \\ b^i + b^\pm = 0, \quad \phi^i_z \mp E^{-\frac{1}{2}} \phi^\pm_\zeta = 0, \quad v^i + v^\pm = v_\pm(r) \\ AT^i + CT^i_z \mp CE^{-\frac{1}{2}} T^\pm_\zeta = 0 \end{array}\right\} \quad \text{at} \quad \zeta = 0. \qquad \text{(A 11)}$$

The solutions for the Ekman–Hartmann layer are known (Gilman & Benton 1968). The thermal equation (A 6a) is easily solved. Altogether, we have

$$v^\pm = (v_\pm(r) - v^i)\, e^{-\beta\zeta} \cos \gamma\zeta, \qquad \text{(A 12)}$$

$$\psi^\pm = \pm E^{\frac{1}{2}} (v_\pm(r) - v^i)\, e^{-\beta\zeta} [m \cos \gamma\zeta + n \sin \gamma\zeta], \qquad \text{(A 13)}$$

$$b^\pm = \pm E^{\frac{1}{2}} (v_\pm(r) - v^i)\, e^{-\beta\zeta} [-n \cos \gamma\zeta + m \sin \gamma\zeta], \qquad \text{(A 14)}$$

$$\phi^\pm = [-E/2(1+\alpha^4)]\, (v_\pm(r) - v^i)\, e^{-\beta\zeta} [\cos \gamma\zeta + \alpha^2 \sin \gamma\zeta], \qquad \text{(A 15)}$$

$$\begin{aligned} T^\pm = {} & \frac{\sigma\epsilon T^i_r}{(1+\alpha^4)^{\frac{1}{2}}} (v_\pm(r) - v^i)\, e^{-\beta\zeta} [\cos \gamma\zeta + \alpha^2 \sin \gamma\zeta] \\ & \mp \frac{\sigma S E^{\frac{1}{2}}}{4(1+\alpha^4)^{\frac{3}{2}}} \frac{(rv_\pm(r) - rv^i)_r}{r} e^{-\beta\zeta} \{\gamma[(1+\alpha^4)^{\frac{1}{2}} + 2\alpha^2] \cos \gamma\zeta \\ & \qquad + \beta[2\alpha^2 - (1+\alpha^4)^{\frac{1}{2}}] \sin \gamma\zeta\}, \end{aligned} \qquad \text{(A 16)}$$

where

$$\beta, \gamma = [(1+\alpha^4)^{\frac{1}{2}} \pm \alpha^2]^{\frac{1}{2}}, \qquad \text{(A 17)}$$

$$n, m = \left[\frac{(1+\alpha^4)^{\frac{1}{2}} \pm \alpha^2}{4+4\alpha^4}\right]^{\frac{1}{2}}. \qquad \text{(A 18)}$$

These solutions satisfy the conditions on $v^\pm$ and $\psi^\pm_\zeta$. The remaining conditions (A 11) yield the compatibility conditions

$$\left.\begin{array}{ll} \psi^i = mE^{\frac{1}{2}}(\mp v_\pm(r) \pm v^i) & \text{(A 19}a\text{)} \\ b^i = nE^{\frac{1}{2}}(\pm v_\pm(r) \mp v^i), \quad \phi^i = mE^{\frac{1}{2}}(\pm v_\pm(r) \mp v^i) & \text{(A 19}b, c\text{)} \\ AT^i + CT^i_z \pm \dfrac{\sigma\epsilon\gamma C}{2E^{\frac{1}{2}}(1+\alpha^4)^{\frac{1}{2}}} T^i_r (v_\pm - v^i) & \\ \qquad - \dfrac{\sigma S C}{2(1+\alpha^4)} \dfrac{1}{r} (rv_\pm - rv^i)_r = 0 & \text{(A 19}d\text{)} \end{array}\right\} \quad \text{at} \quad z = \tfrac{1}{2} \pm \tfrac{1}{2}.$$

Although we have solved a linear boundary-layer equation for the temperature, we have arrived at a nonlinear compatibility equation (A 19d). However, if we assume

$$\epsilon\sigma \ll E^{\frac{1}{2}}(1+\alpha^3), \qquad \text{(A 20)}$$

the third term is much smaller than the second term and the compatibility condition is linearized. Using (A 16) we see that the constraint (A 10) is satisfied if

$$F\sigma\epsilon T^i \ll 1+\alpha^2, \quad F\sigma S E^{\frac{1}{2}} \ll 1+\alpha^5. \tag{A 21}$$

Further, from (A 19*d*) it may be seen that the interior thermal field satisfies the imposed thermal boundary conditions directly provided that

$$\sigma S \ll 1+\alpha^4. \tag{A 22}$$

We shall assume that conditions (A 20)–(A 22) are satisfied by the flow analysed in the main text.

REFERENCES

Acheson, D. J. & Hide, R. 1973 Hydromagnetics of rotating fluids. *Rep. Prog. Phys.* **36**, 159–221.

Barcilon, V. & Pedlosky, J. 1967*a* Linear theory of rotating stratified fluid motions. *J. Fluid Mech.* **29**, 1–16.

Barcilon, V. & Pedlosky, J. 1967*b* A unified linear theory of homogeneous and stratified rotating fluids. *J. Fluid Mech.* **29**, 609–621.

Barcilon, V. & Pedlosky, J. 1967*c* On the steady motions produced by a stable stratification in a rapidly rotating fluid. *J. Fluid Mech.* **29**, 673–690.

Benton, E. R. & Loper, D. E. 1969 On the spin-up of an electrically conducting fluid. Part 1. The unsteady hydromagnetic Ekman–Hartmann boundary-layer problem. *J. Fluid Mech.* **39**, 561–586.

Carrier, G. F. 1965 Some effects of stratification and geometry in rotating fluids. *J. Fluid Mech.* **23**, 145–172.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability.* Oxford: Clarendon Press.

Duncan, I. B. 1966 Axisymmetric convection between two rotating disks. *J. Fluid Mech.* **24**, 417–449.

Gilman, P. A. & Benton, E. R. 1968 Influence of an axial magnetic field on the steady linear Ekman boundary layer. *Phys. Fluids*, **11**, 2397–2401.

Greenspan, H. P. 1968 *The Theory of Rotating Fluids.* Cambridge University Press.

Greenspan, H. P. & Howard, L. N. 1963 On a time-dependent motion of a rotating fluid. *J. Fluid Mech.* **17**, 385–404.

Homsy, G. M. & Hudson, J. L. 1969 Centrifugally driven thermal convection in a rotating cylinder. *J. Fluid Mech.* **35**, 33–52.

Hsueh, Y. 1969 Buoyant Ekman layer. *Phys. Fluids*, **12**, 1757–1762.

Ingham, D. B. 1969 Magnetohydrodynamic flow in a container. *Phys. Fluids*, **12**, 389–396.

Loper, D. E. & Benton, E. R. 1970 On the spin-up of an electrically conducting fluid. Part 2. Hydromagnetic spin-up between infinite flat insulating plates. *J. Fluid Mech.* **43**, 785–799.

Niimi, H. 1971 Stratified fluid motion between two parallel infinite disks rotating differentially. *J. Phys. Soc. Japan*, **31**, 1567–1571.

Roberts, P. H. & Soward, A. M. 1972 Magnetohydrodynamics of the earth's core. *Ann. Rev. Fluid Mech.* **4**, 117–154.

Sakurai, T. 1969*a* Spin down of a Boussinesq fluid in a circular cylinder. *J. Phys. Soc. Japan*, **26**, 840–848.

Sakurai, T. 1969*b* Spin down problem of rotating stratified fluid in thermally insulated circular cylinders. *J. Fluid Mech.* **37**, 689–699.

Vempaty, S. 1974 Hydromagnetic boundary layers in rotating flows. Ph.D. thesis, Florida State University, Tallahassee.